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Abstract

We present an algorithm which computes the convex hull of a set of n spheres in dimension d in time $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. It is worst-case optimal in three dimensions and in even dimensions. The same method can also be used to compute the convex hull of a set of n homothetic convex objects of $\mathbb{E}^d$. If the complexity of each object is constant, the time needed in the worst case is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

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1 Introduction

We present an algorithm which computes the convex hull of a set of n spheres in dimension d in time $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. It is worst-case optimal in three dimensions and in even dimensions. It can also be used to compute the convex hull of a set of n homothetic convex objects of $\mathbb{E}^d$.

Though the complexity and the computation of the convex hull of a set of points in any dimensions is a problem which has been studied extensively, only a few results about the convex hull of a set of spheres are known. The previous results, which are given below, are only for the case $d = 2$ and 3, and, as far as we know, there were no results about the computation of the convex hull of a set of homothetic objects .

The convex hull of a set of spheres is the smallest convex body that contains the spheres. In two dimensions, the boundary of such a convex hull consists of line segments and arcs of circles. In three dimensions, the convex hull boundary is composed of three different kinds of facets (see Figure 1).

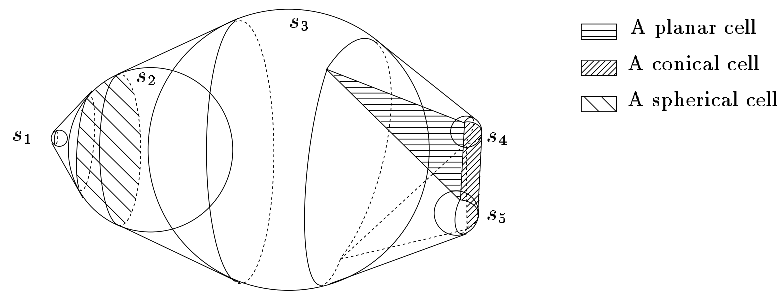

Figure 1: The convex hull of a set of spheres in 3 dimensions

- Planar facets, which are triangles included in planes tangent to three spheres.
- Conical facets, which are parts of cones tangent to two spheres.
- Spherical facets, which are parts of the spheres.

In the plane, the convex hull of a set of disks can be computed in $O(n \log n)$ time (see [6]) which is optimal. In three-dimensional space, the complexity of the convex hull of a set of n spheres is $\Theta(n^2)$ in the worst case, even for collections of pairwise disjoint spheres [7] (see Section 2 below). The convex hull of a set of n spheres in $\mathbb{E}^3$ can be computed in $O(nh)$ time [1], where h is the size of the output (i.e. of the convex hull). In the case where all spheres have the same radius, the convex hull of a set of spheres in $\mathbb{E}^d$ can be easily deduced from the convex hull of the centers of the spheres (see Section 3.1), which can be computed in $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$.

In dimension d, the boundary of the convex hull is composed of d different kinds of facets. Let a *supporting hyperplane of a set* be a hyperplane H which has a non empty intersection with the set and such that the whole set is included in one of the closed halfspaces limited by H. Let a *supporting halfspace of a set* be a halfspace containing the set and limited by a supporting hyperplane of the set. The convex hull of a set of spheres in $\mathbb{E}^d$ is the intersection of the supporting halfspaces of the set of spheres. A *facet of circularity* i $(0 \leq i \leq d-1)$ is a maximal connected portion of the boundary of the convex hull consisting of points where the supporting hyperplanes are tangent to a given set of $(d-i)$ spheres. For example, in dimension 3, the planar facets have circularity 0, the conical facets have circularity 1, the spherical facets have circularity 2.

The boundary of the convex hull of a set of spheres is the union of the closure of facets of circularity $0, 1, 2, ..., d-1$. The boundary of the convex hull is represented by the adjacency graph of these facets.

The paper is organized as follows: In the next Section we give a lower bound on the complexity of the convex hull of a set of n spheres. In Section 3 we present the algorithm to compute this convex hull and we show in Section 4 that it is optimal in three dimensions and in even dimensions. In Section 5 we extend our results to homothetic convex objects.

2 Lower Bounds

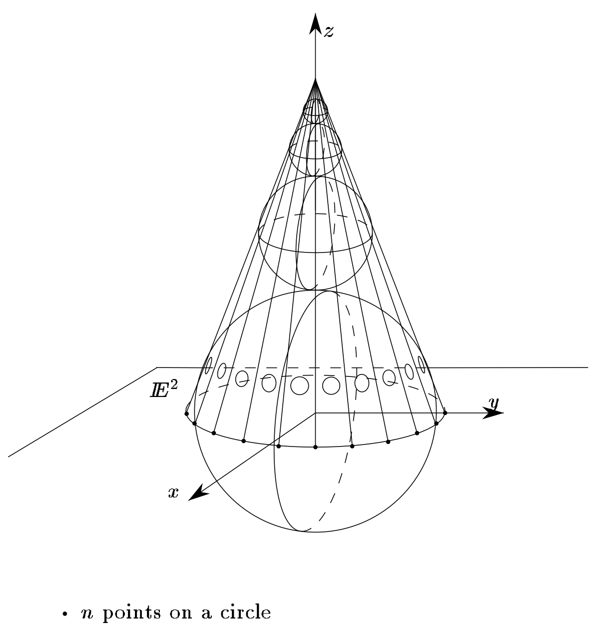

Figure 2: A set of spheres whose convex hull has size $\Theta(n^2)$

In dimension 3, let us take n points, considered as spheres of radius 0, on a circle in the (x, y)-plane and take a point above this plane, on the z-axis. The convex hull of these $n+1$ points is a pyramid. Now add n spheres having non-zero radii and centered on the z-axis, such that each sphere intersects each facet of this pyramid but none of its edges (see Figure 2). The complexity of the convex hull of this set of $2n+1$ spheres is $\Omega(n^2)$.

By the upper bound theorem [5], the complexity of the convex hull of a set of n points in dimension d is $O(n^{\lfloor \frac{d}{2} \rfloor})$ in the worst case. This bound is tight for cyclic polytopes. A point can be considered as a sphere of radius 0. Therefore, the complexity of the convex hull of a set of n spheres is at least equal to the complexity of the convex hull of a set of points, thus is $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$.

We conjecture that the complexity of the convex hull of a set of n spheres is $\Omega(n^{\lceil \frac{d}{2} \rceil})$.

3 The Algorithm

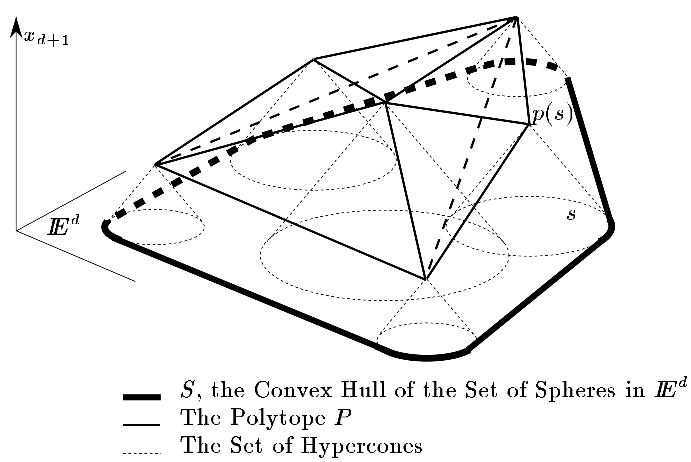

Figure 3: Embeding the spheres in $\mathbb{E}^{d+1}$

We first introduce some notations, then recall some of the properties of duality, and finally give the algorithm that computes the convex hull of a set of spheres.

3.1 Notations and Preliminaries

Let S be the *convex hull of a set of* n *spheres* $\{s_1, ..., s_n\}$ in $\mathbb{E}^d$. We embed $\mathbb{E}^d$ in $\mathbb{E}^{d+1}$ so that the hyperplane $\{x_{d+1} = 0\}$ of $\mathbb{E}^{d+1}$ contains all the spheres. The $(d+1)$-th axis will be called the vertical axis, and in the sequel, the expression *above* will refer to the $(d+1)$-th coordinate. Let s be a sphere in $\mathbb{E}^d$ with center $(x_1, ..., x_d)$ and radius r. Let p be the mapping that associates to s the *point* $p(s)$ in $\mathbb{E}^{d+1}$ such that:

$$p : s \longrightarrow p(s) = (x_1, ..., x_d, r)$$

Let P be the *convex hull of the set of points* $\{p(s_1), ..., p(s_n)\}$ of $\mathbb{E}^{d+1}$. Let λ_0 be a half lower hypercone with arbitrary apex, vertical axis and angle at the apex $\pi/4$.

For a sphere s in $\mathbb{E}^d$, let $\lambda(s)$ be the translated copy of λ_0, with apex $p(s)$. Notice that the intersection between the hypercone $\lambda(s)$ and the hyperplane $\{x_{d+1} = 0\}$ is identical to the sphere s. Let Λ be the *convex hull of the set* $\{\lambda(s_1), ..., \lambda(s_n)\}$ of the n half lower hypercones of $\mathbb{E}^{d+1}$ associated to the n spheres $s_1, ..., s_n$ (see Figure 3). The intersection between Λ and the hyperplane $\{x_{d+1} = 0\}$ is equal to S.

Let O' be a point inside P.

Theorem 1 *Any hyperplane of* $\mathbb{E}^d$ *supporting* S *is the intersection with* $\{x_{d+1} = 0\}$ *of a unique hyperplane* H *of* $\mathbb{E}^{d+1}$ *satisfying the three properties:*

1. H *supports* P*,*

2. H *is the translated copy of a hyperplane tangent to* λ_0 *along one of its generatrices,*

3. H *is above* O'*.*

Conversely, let H *be a hyperplane of* $\mathbb{E}^{d+1}$ *satisfying the above three properties. Its intersection with the hyperplane* $\{x_{d+1} = 0\}$ *is a hyperplane of* $\mathbb{E}^d$ *supporting* S*.*

Proof: Through any point of S of circularity i passes a hyperplane H which supports Λ along a generatrix of at least $d - i$ of the hypercones $\lambda(s_1), ..., \lambda(s_n)$.

This means that H supports P and is the translated copy of a hyperplane tangent to λ_0. As H supports Λ, it is above O'.

Conversely, if an hyperplane H supports P and is above O', it is above P. As H is also the translated copy of a hyperplane tangent to λ_0, it supports Λ, along a generatrix of at least one of the hypercones $\lambda(s_1), ..., \lambda(s_n)$. Its intersection with $\{x_{d+1} = 0\}$ is a hyperplane of $\mathbb{E}^d$ supporting S. $\square$

In the case where all spheres have the same radius, it is easy to see that the convex hull of a set of spheres in $\mathbb{E}^d$ can be obtained by *growing the faces* of the convex hull of the centers of the spheres, i.e. the convex hull of the spheres is exactly the Minkowski Sum of the convex hull of the centers and of a sphere of radius r. Notice that in this case, all the apices of the cones lie on the same horizontal plane $\{x_{d+1} = r\}$, and the growing mechanism can be interpreted as sweeping a plane $\{x_{d+1} = t\}$, t varying from r to 0. Therefore, the complexity drops to $O(n^{\lfloor \frac{d}{2} \rfloor})$ and the running time to $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$.

3.2 Duality

We use *duality* to convert properties 1, 2 and 3 of the above theorem into simpler ones. Let us recall that O' is a point inside P. Let O' be the origin of a new coordinate system, whose axis are parallel to the axis of the previous coordinate system. New coordinates are denoted with a prime: $X' = (x'_1, ..., x'_{d+1})$. Polarity with respect to O' is a one-to-one transformation which maps points of $\mathbb{E}^{d+1}$ distinct from O' to hyperplanes of

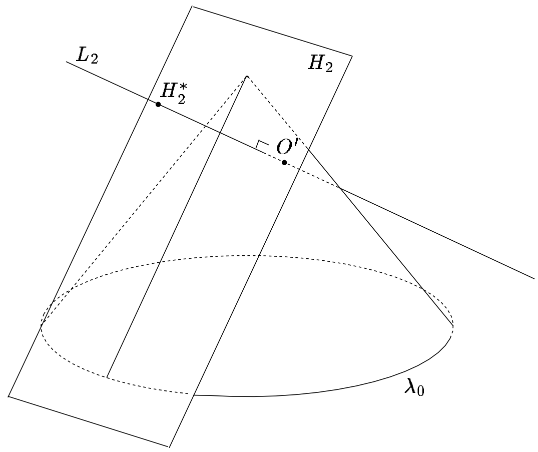

Figure 4: Pole of a hyperplane

$\mathbb{E}^{d+1}$ which do not contain O'. Let M be a point of $\mathbb{E}^{d+1}$ distinct from O'. M^*, the polar hyperplane of M, is defined by the following relation:

$$M^* = \{X' \in \mathbb{E}^{d+1} \;\mid\; M.X' = 1\}$$

H^*, the pole of a hyperplane H not containing O', is defined by

$$H^*.X' = 1, \quad \forall X' \in H$$

We have $(M^*)^* = M$ and $(H^*)^* = H$. Let H^{*-} be the halfspace bounded by H and containing O'.

Let the *polar set* of a set of hyperplanes be the set of poles of these hyperplanes.

Proposition

1. The polytope $P^* = p(s_1)^{*-} \cap ... \cap p(s_n)^{*-}$ of $\mathbb{E}^{d+1}$ is dual to the polytope P, i.e. there is a bijection between the l-faces of P and the $(d-l)$-faces of P^* which reverses the relation of inclusion. Each hyperplane supporting P along a l-face F has its polar point on the corrresponding $(d-l)$-face of F^* of P^*.
2. The polar set of the hyperplanes which are translated copies of the hyperplanes tangent to λ_0 is a hypercone K with apex O', a vertical axis, and an angle at the apex equal to $\pi/4$.
3. The polar set of the hyperplanes above O' is the half space $x'_{d+1} > 0$.

Proof:
The first assertion is well known.

Second assertion: Let H_2 be a hyperplane tangent to λ_0 (see Figure 4). H_2^*, the pole of H_2, belongs to the line L_2 issued from O' and normal to H_2. The polar set of the hyperplanes parallel to H_2 is L_2. The angle of H_2 with the vertical axis is $\pi/4$. Therefore, the angle between L_2 and the vertical axis is $\pi/4$. In order for H_2^* to be well defined, O' may be anywhere inside or outside the hypercone, but not on the hypercone.

As H_2 moves around the hypercone λ_0, staying tangent to it, L_2 moves on a hypercone K with apex O', axis $O'x'_{d+1}$, angle at the apex $\pi/4$. K is a hypersurface of $\mathbb{E}^{d+1}$. The polar set of all the hyperplanes tangent to λ_0 and of the translated copies of these hyperplanes is the hypercone K.

Third assertion: Let H_3 be a hyperplane which lies above O'. Its equation is $\sum_{i=1}^{d+1} h'_i x'_i = 1$. Its intersection with the vertical axis, x'_{d+1}, is such that $h'_{d+1}x'_{d+1} = 1$. As this intersection is above O', we have $x'_{d+1} > 0$ and thus $h'_{d+1} > 0$. As the coordinates of H_3^* are $(h'_1, ..., h'_{d+1})$, H_3^* is in the halfspace $\{x'_{d+1} > 0\}$. Hence, the polar set of the hyperplanes lying above O' is the half space $\{x'_{d+1} > 0\}$. $\square$

P^*	S
$\mathbb{E}^{d+1}$	$\mathbb{E}^d$
i-face $(1 \le i \le d)$	facet of circularity $(d-i)$ $(0 \le d-i \le d-1)$
1-face	facet of circularity $d-1$=part of a sphere
d-face	facet of circularity 0=planar facet

Table 1: correspondence between faces of P and facets of S

The polar set of the hyperplanes supporting the convex hull of the set of points P, tangent to at least one hypercone of Λ along a generatrix and above O' is

$$P^* \cap K \cap \{x'_{d+1} > 0\}$$

Let us consider the intersection of a i-face of P^* with K. The polar hyperplane of each point of this intersection supports P along a $(d-i)$-face and Λ along $(d-i)$ generatrices of $(d-i)$ hypercones. The polar set of this intersection is a family of hyperplanes whose intersection with $\{x_{d+1}=0\}$ is a facet of S of circularity $(d-i)$ (see Table 1).

3.3 The Algorithm

1. Compute the convex hull P and choose a point O' inside P

2. Compute the polytope P^* dual to P with respect to O'.

3. Compute the intersection between P^*, the hypercone K and the half space $\{x'_{d+1} > 0\}$.

4. Compute the incidence graph of the facets of S from the incidence graph of the faces of P^* intersecting K and the halfspace $\{x_{d+1} > 0\}$.

4 Complexity Analysis

Chazelle has shown that the convex hull of a set of n points in dimension $d+1$ can be computed in optimal time $\Theta(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n)$ (see [3]). Simpler randomized algorithms to compute the convex hull of a set of points can be found in [2, 4, 8]. The complexity of computing the polytope P^* dual to P is a linear function of the complexity of P. The complexity of computing the intersection of P^* with K and with $\{x'_{d+1} > 0\}$ is a linear function of the complexity of P^* since K and $\{x'_{d+1} > 0\}$ have constant complexity. The complexity of computing the incidence graph of the facets of S from the incidence graph of the faces of P^* intersecting K and the half space $\{x_{d+1} > 0\}$ is a linear function of the complexity of P^*. Hence, the total amount of time needed to compute S is a linear function of the amount of time needed to compute P. Therefore, the time needed in the worst case to compute S, the convex hull of the set of spheres, is

$$O(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n) = O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$$

This is optimal in three and in even dimensions.

The given reduction from the problem of computing S, the convex hull of a set of spheres in $\mathbb{E}^d$, to that of computing P, the contour of the convex hull of a set of points in $\mathbb{E}^{d+1}$, does not preserve the output size. The complexity of P can exceed that of S. Therefore, using output-sensitive algorithms does not help here.

5 Extension to Homothetic Convex Objects

The algorithm for spheres generalizes to a set of homothetic convex objects having the same orientation. The case of non-convex homothetic objects can be reduced to the case of homothetic convex objects by taking the convex hull of each object. More precisely, let us take a convex object c of $\mathbb{E}^d$ and let c_i $(1 \leq i \leq n)$ be a convex object obtained from c by some homothety and some translation. We compute *the convex hull C of the set of convex objects* $\{c_1, ..., c_n\}$. The main point is that the hypercone K with angle at the apex $\pi/4$ is now replaced by a more *general hypercone* G, which is no longer circular.

Let us associate a half lower hypercone $\lambda(c)$ of $\mathbb{E}^{d+1}$ to c by taking an apex $p(c)$ above the object such that the vertical projection of the apex on $\mathbb{E}^d$ is inside the convex object. $\lambda(c)$ is the half hypercone consisting of the half lines joining $p(c)$ and a point of c. Now we may associate to any object homothetic to c a hypercone $\lambda(c_i)$ which is a translated copy of $\lambda(c)$, such that $\{\lambda(c_i) \cap (x_{d+1} = 0)\} = c_i$,

As before P is the convex hull of $\{p(c_1), ..., p(c_n)\}$ and Λ the convex hull of $\{\lambda(c_1), ..., \lambda(c_n)\}$.

Arguments similar to those of Section 3 can be used. If we replace the half lower hypercone λ_0 by $\lambda(c)$, we have the following theorem:

Theorem 2 *Any hyperplane of $\mathbb{E}^d$ supporting C is the intersection with $\{x_{d+1} = 0\}$ of a unique hyperplane H of $\mathbb{E}^{d+1}$ satisfying the three properties:*

1. *H supports P*
2. *H is the translated copy of a hyperplane tangent to $\lambda(c)$ along one of its generatrices.*
3. *H is above O' .*

Conversely, let H be a hyperplane of $\mathbb{E}^{d+1}$ satisfying the above three properties. Its intersection with the hyperplane $\{x_{d+1} = 0\}$ is a hyperplane of $\mathbb{E}^d$ supporting C.

The dual of the set of hyperplanes H satisfying Condition 2 is now a general hypercone G with apex O', which is no longer circular.

The algorithm of Section 3 can be used if we replace the hypercone K by G.

We assume that the convex objects c have constant complexity. Hence the complexity of the hypercone G is constant. The complexity analysis remains the same as for spheres. Replacing K by G does not change the complexity of the algorithm since G has constant complexity. Therefore, the convex hull of n homothetic convex objects of constant complexity in dimension d can be computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ time.

For example, the time needed to compute the convex hull of n homothetic ellipsoids in dimension d is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

Let us notice that, similarly to the case of spheres having the same radii, the convex hull of a set of translates of a given convex object can be easily computed in $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$.

6 Conclusion

In this paper, we have reduced by a suitable geometric transform the problem of constructing the convex hull of n spheres in d-space to the problem of computing the intersection of a $(d+1)$-polytope with n facets with a hypercone. We have shown that the convex hull of n spheres in dimension d can be computed in $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ time in the worst case, which is optimal in dimension 3 and in even dimensions. We conjecture that the algorithm is optimal in all dimensions.

We have extended these results to homothetic convex objects: If each object has constant complexity, the time needed in the worst case to compute their convex hull is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. Computing the convex hull of general ellipsoids or convex objects in dimension $d \geq 3$ remains an open problem.

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